Lineaire Algebra II
11/04/2011

I a) 2 No, because $\langle u,v\rangle = 3/5 \neq 0$

$\langle v,w\rangle = \frac{12}{25} \neq 0$

$\langle w,u\rangle = 4/5 \neq 0$

To be orthogonal all inner products with all vectors have to be zero.

b) 3 $\|5u-3v\| = (\langle 5u-3v, 5u-3v\rangle)^{1/2}$

$= (\langle 5u,5u\rangle + \langle -3v,-3v\rangle + 2\langle -3v,5u\rangle)^{1/2}$

$= (25\langle u,u\rangle + 9\langle v,v\rangle \cancel{+} -30\langle v,u\rangle)^{1/2}$

$= (25\|u\|^2 + 9\|v\|^2 - 30\cdot\frac{3}{5})^{1/2}$

$= (25 + 9 - 18)^{1/2}$

1) $= (16)^{1/2} = 4$

c) Orthonormal basis for span$(\{u,v,w\})$

$u_1 = \frac{1}{\|x_1\|} x_1$ with $x_1 = u$

$u_1 = u$

$u_2 = \frac{1}{\|x_2 - p_1\|}(x_2 - p_1)$ with $x_2 = v$

$p_1 = \langle x_2, u_1\rangle u_1$

$= \langle v, u_1\rangle u_1$

$= \frac{3}{5} u$

$u_2 = \frac{1}{\|v - \frac{3}{5}u\|}(v - \frac{3}{5}u)$

$\|v - \frac{3}{5}u\| = (\langle v - \frac{3}{5}u, v - \frac{3}{5}u\rangle)^{1/2}$

$= (\langle v,v\rangle + \frac{9}{25}\langle u,u\rangle + 2\langle -\frac{3}{5}u, v\rangle)^{1/2}$

$= (1 + \frac{9}{25} \cancel{+} - \frac{6}{5}\frac{3}{5})^{1/2}$

$= \left(\frac{16}{25}\right)^{1/2} = \frac{4}{5}$

$$u_2 = \frac{1}{\|x_2 - p_1\|}(x_2 - p_1)$$

$$= \frac{5}{4}\left(v - \frac{3}{5}u\right)$$

$$u_3 = \frac{1}{\|x_3 - p_2\|}(x_3 - p_2) \qquad x_3 = w$$

$$p_2 = \langle x_3, u_2 \rangle u_2 + \langle x_3, u_1 \rangle u_1$$
$$= \left\langle w, \frac{5}{4}\left(v - \frac{3}{5}u\right)\right\rangle \frac{5}{4}\left(v - \frac{3}{5}u\right) + \langle w, u \rangle u$$

$$= \left(\frac{5}{4}\langle w, v \rangle - \frac{3}{4}\langle w, u \rangle\right)\frac{5}{4}\left(v - \frac{3}{5}u\right) + \langle w, w \rangle u$$

$$= \left(\frac{5}{4} \cdot \frac{12}{25} - \frac{3}{4}\frac{4}{5}\right)\frac{5}{4}\left(v - \frac{3}{5}u\right) + \frac{4}{5}u$$

$$= \underbrace{\left(\frac{12}{20} - \frac{12}{20}\right)}_{0}\frac{5}{4}\left(v - \frac{3}{5}u\right) + \frac{4}{5}u$$

$$= \frac{4}{5}u$$

$$u_3 = \frac{1}{\|w - \frac{4}{5}u\|}\left(w - \frac{4}{5}u\right)$$

$$\left(\left\langle w - \frac{4}{5}u, w - \frac{4}{5}u \right\rangle\right)^{1/2} = \left(\langle w, w \rangle + \frac{16}{25}\langle u, u \rangle - 2 \cdot \frac{4}{5}\langle u, w \rangle\right)^{1/2}$$

$$= \left(1 + \frac{16}{25} - \frac{8}{5} \cdot \frac{4}{5}\right)^{1/2}$$

$$= \left(\frac{25}{25} + \frac{16}{25} - \frac{32}{25}\right)^{1/2}$$

$$= \left(\frac{9}{25}\right)^{1/2} = \frac{3}{5}$$

$$u_3 = \frac{5}{3}\left(w - \frac{4}{5}u\right)$$

Dus orthonormale span van span($\{u, v, w\}$)

$= \text{span}(\{u, \frac{5}{4}(v - \frac{3}{5}u), \frac{5}{3}(w - \frac{4}{5}u)\})$

$\boxed{2}$ $M = \begin{bmatrix} a & 0 \\ b & c \end{bmatrix}$

a) For which values of (a,b,c) is this matrix

2 diagonalizable.

Not the same { M is diagonalizable iff all eigenvalues (eigenvectors) are distinct (linearly independent).

$$\det(M - \lambda I) = \begin{vmatrix} a-\lambda & 0 \\ b & c-\lambda \end{vmatrix} = (a-\lambda)(c-\lambda)$$

$(a-\lambda)(c-\lambda) = 0$ als $\lambda_1 = a$ en $\lambda_2 = c$

M is diagonalizable if $\lambda_1 \neq \lambda_2$ and .....

So for every $a \neq c \in \mathbb{R}$ and for every $b \in \mathbb{R}$ M is diagonalizable.

b) Diagonalizable by a unitary matrix iff the matrix is normal.

So $M^H M = M M^H$

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$$\begin{bmatrix} \bar{a} & \bar{b} \\ 0 & \bar{c} \end{bmatrix}\begin{bmatrix} a & 0 \\ b & c \end{bmatrix} = \begin{bmatrix} a & 0 \\ b & c \end{bmatrix}\begin{bmatrix} \bar{a} & \bar{b} \\ 0 & \bar{c} \end{bmatrix}$$

$$\begin{bmatrix} \bar{a}a + \bar{b}b & \bar{b}c \\ b\bar{c} & \bar{c}c \end{bmatrix} = \begin{bmatrix} a\bar{a} & a\bar{b} \\ \bar{a}b & b\bar{b} + c\bar{c} \end{bmatrix}$$

so $\bar{a}a + \bar{b}b = a\bar{a}$, $\bar{b}c = a\bar{b}$ $b\bar{c} = \bar{a}b$

and $\bar{c}c = b\bar{b} + \bar{c}c$

$\Downarrow$ $\Downarrow$ $\Downarrow$

$b\bar{b} = 0$ $\cancel{c = a}$ since $b=0$ $\cancel{\bar{c} = \bar{a}}$

if $b = x + iy$ with $x, y \in \mathbb{R}$ then $b\bar{b} = (x+iy)(x-iy) = x^2 + y^2 = 0$

$x^2 + y^2 = 0$ for $x, y \in \mathbb{R}$ iff $x = y = 0$ so $b = 0$

So it is diagonalizable by ~~an~~ unitary matrix if $b=0$ and $\cancel{a}=c$

c) Diagonalizable by orthogonal matrix iff $M^T=M$

0

$$\cancel{M^TM=MM^T}$$

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\begin{bmatrix} a & 0 \\ b & c \end{bmatrix}=\begin{bmatrix} a & 0 \\ b & c \end{bmatrix}\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$$

$$\begin{bmatrix} a^2+b^2 & bc \\ bc & c^2 \end{bmatrix}=\begin{bmatrix} a^2 & ab \\ ab & b^2+c^2 \end{bmatrix}$$

so $a^2+b^2=a^2$ $\Downarrow$ $b^2=0$ $\quad bc=ab$ $\Downarrow$ $c=a$ $\quad c^2=b^2+c^2$ $\Downarrow$ $b^2=0$

$(x+iy)(x+iy)=x^2-y^2+2xyi=0$ with $x,y\in\mathbb{R}$ $\Rightarrow$ so $x=y=0$

and again we have $b=0$ and $a=c$

$\boxed{3}$ $A\in\mathbb{R}^{n\times n}$ symmetric matrix

a) 2 A is normal?

Proof: If A is normal then $A^HA=AA^H$

Since A is symmetric and real we have that $A=A^H$. So $A^HA=A^HA^H=AA^H$ so A is normal.

b) 3 All eigenvalues of A are real?

so $Ax\overset{\text{for } x\neq 0}{=}\lambda x$ and $x^HAx=x^H\lambda x=\lambda x^Hx=\lambda\overset{>0}{\|x\|^2}$

Since A is Hermitian we also have that:

$\lambda\|x\|^2=x^HAx=x^HA^Hx=(Ax)^Hx=(\lambda x)^Hx=x^H\lambda^Hx=\lambda^H x^Hx=\lambda^H\|x\|^2$

So we have that $\lambda\|x\|^2=\lambda^H\|x\|^2$

continued

3 b) So $\lambda\|x\|^2 = \lambda^H\|x\|^2$

Since because $\|x\|^2 > 0$ (because $x \neq 0$)

we must have that $\lambda = \lambda^H = \bar{\lambda}$

Since $\lambda = \bar{\lambda}$, this means that λ is real.

c) 3 $A^H A x = A^T A x = \omega x$ $\quad$ ω is eigenvalue of $A^T A$ $\quad$ $Ax = \lambda x$

And $x^H A^H A x = \|Ax\|^2 = \|\lambda x\|^2 = |\lambda|^2 \|x\|^2$

But $x^H A^H A x = x^H A^T A x = x^H \omega x = \omega x^H x = \omega \|x\|^2$

So $\omega = |\lambda|^2$.

But a singular value of A is given by $\sigma = \sqrt{\omega}$

so $\sigma = \sqrt{\omega} = \sqrt{|\lambda|^2} = |\lambda|$.

So $|\lambda|$ is a singular value of A if λ is an eigenvalue of A.

d) 0

$\cancel{A = U\Sigma V^T}$ $\quad$ $\cancel{\text{let } |\lambda_1| \geq |\lambda_2| \geq \dots \geq |\lambda_k|}$

Since A is Hermitian we have that:

$A = UDU^H$ $\quad$ with λ's on Diagonal of D.

with $UU^H = I$

Also $AA^H = UDU^H(UD^H U) = UDD^H U$.

But $AA^H = A^H A = A^2$ (symmetric).

Further on now: $A^2 = UDD^H U$. Because D has all eigenvalues of A on the diagonal (which are positive or negative) multiplying with D^H gives only λ^2 on diagonal.
Now A becomes $A = (UDD^H U)^{1/2} = U^{1/2} D^{1/2} (D^H)^{1/2} U^{1/2}$

~~d) Best Rank k approximation of A.~~

Since $D^{1/2}(D^H)^{1/2}$ has got $|\lambda|$'s on diagonal this becomes our Σ. Further on $U^{1/2}$ is our orthogonal diagonalizer because $(U^{1/2})(U^{T})^{1/2} = (UU^T)^{1/2}$
$= I$

In the above answer all H's have to be changed by T signs so: $\underline{U^H = U^T}$.

Now our SVD becomes: $A = U^{1/2} \Sigma U^{1/2}$

e)[2] Instead of Σ with n ~~eigenvalues~~ singular values on the diagonal, you have to leave k singular values on Σ for $k' > k$ all diagonal elements are 0 so

$$A = U^{1/2} \Sigma U^{1/2}$$

$$A_{(k)} = U^{1/2} \begin{pmatrix} \Sigma_k & 0 \\ 0 & 0 \end{pmatrix} U^{1/2}$$

k^{th} approximation

$$= U_k^{1/2} \Sigma_k U_k^{1/2}$$

4] a) $f(x,y) = \sin(x) + y^3 + 3xy + 2x - 3y$

i) 2 $(0,-1)$ stationary point?

$$f_x = \cos x + 3y + 2$$
$$f_y = 3y^2 + 3x - 3$$

$$\left.\begin{array}{l} f_x(0,-1) = 1 - 3 + 2 = 0 \\ f_y(0,-1) = 3 + 0 - 3 = 0 \end{array}\right\}$$ so $f_x = f_y = 0$ So indeed stationary point.

(ii) 6 Hessian matrix: $\left.\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}\right|_{(0,-1)}$

$$\left.\begin{array}{l} f_{xx} = -\sin x \\ f_{yy} = 6y \\ f_{xy} = 3 \\ f_{yx} = 3 \end{array}\right\} H = \left.\begin{bmatrix} -\sin x & 3 \\ 3 & 6y \end{bmatrix}\right|_{(0,-1)}$$

$$= \begin{bmatrix} 0 & 3 \\ 3 & -6 \end{bmatrix}$$

Calculating eigenvalues:

$$\begin{vmatrix} -\lambda & 3 \\ 3 & -6-\lambda \end{vmatrix} = \lambda(6+\lambda) - 9 = 0$$

$$\lambda^2 + 6\lambda - 9 = 0$$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-6 \pm \sqrt{36+36}}{2} = \frac{-6 \pm \sqrt{9 \cdot 8}}{2}$$
$$= \frac{-6 \pm 3\sqrt{8}}{2}$$

(indefinite)

indefinite $\left\{\begin{array}{l} \lambda_1 = \frac{-6 - 3\sqrt{8}}{2} < 0 \\ \lambda_2 = \frac{-6 + 3\sqrt{8}}{2} > 0 \end{array}\right\}$ So because signs are not equal $(0,-1)$ is a saddle point

4b) 6 A is symmetric matrix so $A^T = A$

Since $e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \dots$

$$(e^A)^T = \left(I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \dots\right)^T$$

$$= \left(I^T + A^T + \frac{(A^2)^T}{2!} + \frac{(A^3)^T}{3!} + \dots\right)$$

$A = A^T \longrightarrow$

$$= \left(I + A + \frac{(A^T)^2}{2!} + \frac{(A^T)^3}{3!} + \dots\right)$$

$$= I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \dots$$

$$= e^A$$

So $e^A = (e^A)^T$ so e^A is symmetric

Assuming you mean a symmetric real matrix we have $A^H = A^T = A$. So A can be diagonalized with a unitary matrix.

$$A = UDU^H$$

$$e^A = e^{UDU^H} = I + UDU^H + \frac{(UDU^H)^2}{2!} + \dots$$

$$= U\left(I + D + \frac{D^2}{2!} + \dots\right)U^H$$

$$= U e^D U^H$$

~~Since~~

$$= U\begin{pmatrix} e^{\lambda_1} & & \\ & e^{\lambda_2} & \\ & & \ddots \; e^{\lambda_n} \end{pmatrix}U^H$$

Since e^λ is always greater than zero and because e^λ are all on the diagonal

wht?

continued

4b we have that (e^{λ_i})'s are the eigenvalues of the matrix e^A. Since e^{λ_i} is always greater than zero we have that all eigenvalues are greater than zero. So e^A is positive definite.

5] $A \in \mathbb{R}^{2\times 2}$ $\quad p_A(\lambda) = \lambda^2 - \lambda - 1$

$\alpha_0 = \alpha_1 = 1$

and $\alpha_{k+2} = \alpha_{k+1} + \alpha_k$ for $n \geq 0$

Show that $A^{n+2} = \alpha_{n+1} A + \alpha_n I$

for $n \geq 0$.

First prove for $n=0$
then $A^2 = \alpha_1 A + \alpha_0 I = A + I$ =
Because $p(\lambda) = \lambda^2 - \lambda - 1$ so $p(A) = A^2 - A - I = 0$
So $A^2 = A + I$. So for $n=0$ the hypothesis = holds.

Now assume it holds for all k. Then to prove the hypothesis we have to prove that it also holds for $k+1$.

For k
This holds: $A^{(k)+2} = \alpha_{(k+1)} A + \alpha_k I$

Multiplying both sides with A gives us.

$$A^{(k+1)+2} = \alpha_{k+1} A^2 + \alpha_k A \qquad \boxed{A^2 = A + I}$$

$$= \alpha_{k+1}(A+I) + \alpha_k A$$

$$= (\alpha_{k+1} + \alpha_k) A + \alpha_{k+1} I$$

Since $\alpha_{k+1} + \alpha_k = \alpha_{k+2}$ we get

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$$A^{(k+1)+2} = \alpha_{k+2} A + \alpha_{k+1} I$$

So the statement holds also for $k+1$ and thus

$$A^{n+2} = \alpha_{n+1} A + \alpha_n I \text{ holds for all } n \geq 0.$$

$\boxed{6}$ $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

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Eigenvalues $\Rightarrow$ $\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 & 3 \\ 0 & 1-\lambda & 2 \\ 0 & 0 & 1-\lambda \end{vmatrix}$

$= (1-\lambda)^3 = 0$

$\lambda = 1$ with multiplicity 3

Eigenvectors: $N(A - \lambda I)$

$$\begin{bmatrix} 0 & 2 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$0x_1 + 2x_2 + 3x_3 = 0 \Rightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$2x_3 = 0 \Rightarrow x_3 = 0$

$2x_2 + 3x_3 = 0$

$x_2 = -\frac{3}{2} x_3$ so $x_2 = 0$

So $x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is only eigenvector.

Because the number of jordan blocks is determined by the number of eigenvector, we have that J has 1 block

So $J = \begin{bmatrix}1&1&0\\0&1&1\\0&0&1\end{bmatrix}$

Now the highest number k such that it is still consistent:

$$(A-\lambda I)^k s = x$$

$$(A-I)^2 = \begin{bmatrix}0&2&3\\0&0&2\\0&0&0\end{bmatrix}\begin{bmatrix}0&2&3\\0&0&2\\0&0&0\end{bmatrix} = \begin{bmatrix}0&0&\overset{4}{\cancel{12}}\\0&0&0\\0&0&0\end{bmatrix}$$ is consistent

$$(A-I)^3 = \begin{bmatrix}0&0&12\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}0&2&3\\0&0&2\\0&0&0\end{bmatrix} = \begin{bmatrix}0&0&\overset{0}{\cancel{24}}\\0&0&0\\0&0&0\end{bmatrix}$$

$(A-I)^3$ and $(A-I)^2$ are equal up to a constant.

Now we can use $(A-I)^2$:

$$(A-I)^2 s = x$$

$$\begin{bmatrix}0&0&12\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 12 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} 12\, x_3 &= 1 \\ x_3 &= \tfrac{1}{12} \end{aligned}\right\} \quad x_2 = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{12} \end{bmatrix}$$

$$(A-I)x_2 = \begin{bmatrix} 0 & 2 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \frac{1}{12} \end{bmatrix} = \begin{bmatrix} \frac{3}{12} \\ \frac{2}{12} \\ 0 \end{bmatrix} = x_3$$

$$\text{Dus } X = [x \;\; x_3 \;\; x_2] = \begin{bmatrix} 1 & \frac{3}{12} & 0 \\ 0 & \frac{2}{12} & 0 \\ 0 & 0 & \frac{1}{12} \end{bmatrix}$$